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PT symmetry and large-N models

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Abstract

Recently developed methods for PT-symmetric models can be applied to quantum-mechanical matrix and vector models. In matrix models, the calculation of all singlet wavefunctions can be reduced to the solution of a one-dimensional PT-symmetric model. The large-N limit of a wide class of matrix models exists, and properties of the lowest-lying singlet state can be computed using WKB. For models with cubic and quartic interactions, the ground-state energy appears to show rapid convergence to the large-N limit. For the special case of a quartic model, we find explicitly an isospectral Hermitian matrix model. The Hermitian form for a vector model with O(N) symmetry can also be found, and shows many unusual features. The effective potential obtained in the large-N limit of the Hermitian form is shown to be identical to the form obtained from the original PT-symmetric model using familiar constraint field methods. The analogous constraint field prescription in four dimensions suggests that PT-symmetric scalar field theories are asymptotically free.

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1. Introduction

Since the initial discovery of PT symmetry [1], there has been considerable progress in expanding both the number of PT-symmetric models and our knowledge of their properties [2, 3]. However, models with continuous internal symmetry groups have not been extensively developed. Of course, the field theories relevant to modern particle physics have continuous symmetries, and it is natural to seek PT-symmetric models with similar continuous symmetries. Here we review recent progress we have made in the construction and analysis of PT-symmetric models of scalars with $O(N)$ or $U(N)$ symmetry [4, 5]. Most of the results will deal with quantum-mechanical models, usefully regarded as one-dimensional field theories. Of particular interest is the construction of the large-N limit, as this has proven to be a very powerful theoretical tool in the analysis of many different field theories.

Hermitian matrix models appear in many contexts in modern theoretical physics, with applications ranging from condensed matter physics to string theory. Interest in the large-N limit of matrix models was strongly motivated by work on the large-N_c limit of QCD [6], but interest today is much wider. For example, Hermitian matrix quantum mechanics leads to a construction of two-dimensional quantum gravity coupled to $c = 1$ matter [7]. It is surprising that the construction of PT-symmetric matrix models is somewhat easier than the construction of models with a vector symmetry. The matrix techniques pioneered in [8] for Hermitian matrix quantum mechanics can be extended to PT-symmetric matrix quantum mechanics. In these models, the contours of functional integration over the matrix eigenvalues are extended into the complex plane. The large-N limit can then be taken in PT-symmetric matrix theories just as in the Hermitian case. Quantities of interest such as the scaled ground-state energy and scaled moments can be calculated using WKB methods. In the special case of a quartic potential with the 'wrong' sign, we use functional integration techniques to prove that the PT-symmetric model is equivalent to a Hermitian matrix model with an anomaly for all values of N, as in the one-component case [9, 10]. Interestingly, the anomaly vanishes to leading order in the large-N limit.

Although the construction of PT-symmetric matrix models has proved to be relatively straightforward, the construction of PT-symmetric models with fields transforming as vectors under $O(N)$ is more difficult technically. Nevertheless, the development of scalar field theory models with vector symmetry is crucial to the possible relevance of PT-symmetry in particle physics. Only models with quartic interactions have so far proved tractable. This progress on quartic models with N components is built upon recent work on the relation of the one-component $-\lambda x^4$ model to its equivalent Hermitian form [9, 10], as well as recent work on the relation of $O(N)$-symmetric Hermitian models to one-component PT-symmetric models [11]. As in the single-component and matrix cases, the PT-symmetric model with $O(N)$ symmetry and quartic interaction also proves to have a Hermitian form for all values of N. The Hermitian form of the PT-symmetric $O(N)$ model allows a technically straightforward construction of the large-N limit, which can in turn be compared with simpler methods that lead to essentially the same result at leading order. The constraint-field method is particularly notable, apart from its simplicity and familiarity, because it generalizes to provide the form for the effective potential of PT-symmetric scalar field theories in the large-N limit. This effective potential in turn implies asymptotic freedom in four dimensions, a property long suspected to hold in a renormalizable PT-symmetric scalar field theory in four dimensions.

This paper is organized as follows: section 2 develops the formalism required to treat PT-symmetric matrix models, and section 3 analyzes the properties of the ground-state of such models using WKB methods. Section 4 treats the special case of the $\mathrm{Tr}\, M^4$ model, which has a simple Hermitian dual representation. In section 5, a large class of PT-symmetric models with N components are described, including vector models with $O(N)$ symmetry. Section 6 shows that models in this class have simple Hermitian dual representations. In section 7, the large-N limit of the $O(N)$ model is derived using three different methods, while section 8 applies one of these methods to PT-symmetric scalar field theories in the large-N limit.

2. Formalism for matrix models

The techniques for solving Hermitian matrix models are well known. The solution for all N of the quantum mechanics problem associated with the Euclidean Lagrangian

$$L = \frac{1}{2}\mathrm{Tr}\left(\frac{\mathrm{d}M}{\mathrm{d}t}\right)^2 + \frac{g}{N}\mathrm{Tr}\, M^4 \tag{1}$$

where M is an $N \times N$ Hermitian matrix was first given by Brezin *et al* [8]. The ground state ψ is a symmetric function of the eigenvalues λ_j of M. The antisymmetric wavefunction ϕ defined by

$$\phi(\lambda_1, \ldots, \lambda_N) = \left[\prod_{j<k}(\lambda_j - \lambda_k)\right] \psi(\lambda_1, \ldots, \lambda_N) \tag{2}$$

satisfies the Schrödinger equation

$$\sum_j \left[-\frac{1}{2}\frac{\partial^2}{\partial \lambda_j^2} + \frac{g}{N}\lambda_j^4 \right] \phi = N^2 E^{(0)} \phi, \tag{3}$$

where $E^{(0)}$ is the ground-state energy scaled for the large-N limit. This equation separates into N individual Schrödinger equations, one for each eigenvalue, and the antisymmetry of ϕ determines $N^2 E^{(0)}$ as the sum of the N lowest eigenvalues.

Here we solve the corresponding problem where the potential term is PT-symmetric but not Hermitian. As shown by Bender and Boettcher [1], the one-variable problem may be solved by extending the coordinate variable into the complex plane. This implies that for PT-symmetric matrix problems, we must analytically continue the eigenvalues of M into the complex plane, and in general M will be normal rather than Hermitian. We consider the Euclidean Lagrangian

$$L = \frac{1}{2}\,\mathrm{Tr}\left(\frac{\mathrm{d}M}{\mathrm{d}t}\right)^2 - \frac{g}{N^{p/2-1}}\,\mathrm{Tr}\,(\mathrm{i}M)^p \tag{4}$$

with $g > 0$. Making the substitution $M \to U\Lambda U^+$, with U unitary and Λ diagonal, we can write L as

$$L = \frac{1}{2}\sum_j \left(\frac{\mathrm{d}\lambda_j}{\mathrm{d}t}\right)^2 + \sum_{j,k}\frac{1}{2}(\lambda_j - \lambda_k)^2 \left(\frac{\mathrm{d}H}{\mathrm{d}t}\right)_{jk}\left(\frac{\mathrm{d}H}{\mathrm{d}t}\right)_{kj} - \frac{g}{N^{p/2-1}}\sum_j (\mathrm{i}\lambda_j)^p, \tag{5}$$

where

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -\mathrm{i}U^+\frac{\mathrm{d}U}{\mathrm{d}t}. \tag{6}$$

In the analysis of conventional matrix models by Brezin *et al*, a variational argument shows that the ground state is a singlet, with no dependence on U. Because the λ_j's are in general complex for PT-symmetric theories, this argument does not apply. However, in two cases we can prove that the ground state is indeed a singlet: for $p = 2$, which is trivial, and for $p = 4$, where the explicit equivalence with a Hermitian matrix model proven below can be used. Henceforth, we will assume that the ground state is a singlet, but our results will apply in any case to the lowest-energy singlet state.

We have now reduced the problem of finding the ground state to the problem of solving for the first N states of the single-variable Hamiltonian

$$H = \frac{1}{2}p^2 - \frac{g}{N^{p/2-1}}(\mathrm{i}\lambda)^p. \tag{7}$$

This Hamiltonian is PT-symmetric but in general not Hermitian. The case $p = 2$ is the simple harmonic oscillator. For $p > 2$, the Schrödinger equation associated with each eigenvalue may be continued into the complex plane as explained in [1]. We exclude the case $p < 2$, where PT symmetry is spontaneously broken and the eigenvalues of H are no longer real.

3. Ground-state properties of matrix models

As with Hermitian matrix models, the ground-state energy is the sum of the first N eigenenergies of the Hamiltonian H. In the large-N limit, this sum may be calculated using WKB. A novelty of WKB for PT-symmetric models is the extension of classical paths into the complex plane. This topic has been treated extensively for single-component models [1, 12].

We define the Fermi energy E_F as the energy of the Nth state,

$$N = \frac{1}{2\pi} \int \mathrm{d}p \, \mathrm{d}\lambda \, \theta[E_F - H(p, \lambda)], \tag{8}$$

where the path of integration must be a closed, classical path in the complex p–λ plane. In order to construct the large-N limit, we perform the rescaling $p \to \sqrt{N} p$ and $\lambda \to \sqrt{N} \lambda$ yielding

$$H_{sc}(p, \lambda) = \tfrac{1}{2} p^2 - g(\mathrm{i}\lambda)^p, \tag{9}$$

where the scaled Hamiltonian H_{sc} is related to H by $H = N H_{sc}$. We introduce a rescaled Fermi energy ϵ_F given by $E_F = N\epsilon_F$, which is implicitly defined by

$$1 = \frac{1}{2\pi} \int \mathrm{d}p \, \mathrm{d}\lambda \, \theta \left[\epsilon_F - H_{sc}(p, \lambda)\right]. \tag{10}$$

After carrying out the integration over p, we have

$$1 = \frac{1}{\pi} \int \mathrm{d}\lambda \, \sqrt{2\epsilon_F + 2g(\mathrm{i}\lambda)^p} \theta[\epsilon_F + g(\mathrm{i}\lambda)^p] \tag{11}$$

where the contour of integration is taken along a path between the turning points which are the analytic continuation of the turning points at $p = 2$. This equation determines ϵ_F as a function of g.

We define a scaled ground-state energy $E^{(0)}$ by

$$E_N^{(0)} = \frac{1}{N^2} \sum_{k=0}^{N-1} E_k. \tag{12}$$

The WKB result for the sum of the energies less than E_F can be written as

$$\sum_{k=0}^{N-1} E_k = \frac{N^2}{2\pi} \int \mathrm{d}p \, \mathrm{d}\lambda \, H_{sc}(p, \lambda) \theta[\epsilon_F - H_{sc}(p, \lambda)] \tag{13}$$

so that in the large-N limit $E_\infty^{(0)}$ is given by

$$E_\infty^{(0)} = \frac{1}{2\pi} \int \mathrm{d}p \, \mathrm{d}\lambda \, H_{sc}(p, \lambda) \theta[\epsilon_F - H_{sc}(p, \lambda)]. \tag{14}$$

The integration over p is facilitated by using equation (10) to insert a factor of ϵ_F, giving

$$E_\infty^{(0)} = \epsilon_F - \frac{1}{2\pi} \int \mathrm{d}p \, \mathrm{d}\lambda [\epsilon_F - H_{sc}(p, \lambda)] \theta[\epsilon_F - H_{sc}(p, \lambda)]. \tag{15}$$

The integral over p then yields

$$E_\infty^{(0)} = \epsilon_F - \frac{1}{3\pi} \int \mathrm{d}\lambda [2\epsilon_F + 2g(\mathrm{i}\lambda)^p]^{3/2} \theta[\epsilon_F + g(\mathrm{i}\lambda)^p]. \tag{16}$$

The turning points in the complex λ plane are

$$\lambda_- = \left(\frac{\epsilon_F}{g}\right)^{1/p} \mathrm{e}^{\mathrm{i}\pi(3/2 - 1/p)} \tag{17}$$

Table 1. The scaled ground-state energy $E_N^{(0)}$ at $g = 1$ for $p = 3$ and 4. The finite N results obtained numerically rapidly approach the $N \to \infty$ limit obtained from WKB.

N	$p = 3$	$p = 4$
g1	0.762 852	0.930 546
2	0.756 058	0.935 067
3	0.75 486	0.935 846
4	0.754 443	0.936 115
5	0.754 251	0.936 239
6	0.754 147	0.936 306
7	0.754 084	0.936 347
8	0.754 043	0.936 372
∞	0.753 991	0.936 458

$$\lambda_+ = \left(\frac{\epsilon_F}{g}\right)^{1/p} \mathrm{e}^{-\mathrm{i}\pi(1/2-1/p)}. \tag{18}$$

We integrate λ along a two-segment, straight-line path connecting the two turning points via the origin [1]. Solving equation (10) for ϵ_F, we find

$$\epsilon_F = \left[\left(\frac{\pi}{2}\right)^p \left(\frac{\Gamma(3/2+1/p)}{\sin(\pi/p)\Gamma(1+1/p)}\right)^{2p} g^2\right]^{\frac{1}{p+2}}, \tag{19}$$

and solving (16) for the scaled ground-state energy we have

$$E_\infty^{(0)} = \frac{p+2}{3p+2}\epsilon_F = \frac{p+2}{3p+2}\left[\left(\frac{\pi}{2}\right)^p \left(\frac{\Gamma(3/2+1/p)}{\sin(\pi/p)\Gamma(1+1/p)}\right)^{2p} g^2\right]^{\frac{1}{p+2}}. \tag{20}$$

For $p = 2$, this evaluates to $E^{(0)} = \sqrt{g/2}$, in agreement with the explicit result for the harmonic oscillator.

It is very interesting to compare the large-N result with results for finite N. The low-lying eigenvalues for the Hamiltonian $p^2 - (\mathrm{i}x)^p$ have been calculated by Bender and Boettcher in [1] for the cases $p = 3$ and 4; the case $p = 2$ is trivial. We can use their results by noting that the eigenvalues of our Hamiltonian H are related to theirs by

$$E_j = \frac{g^{2/(p+2)}}{2^{p/(p+2)} N^{(p-2)/(p+2)}} E_j^{BB}. \tag{21}$$

Results for $p = 3$ and 4 and small values of N are compared with the large-N limit in table 1. The energies for finite values of N rapidly approach the $N \to \infty$ limit. The approach to the limit appears monotonic in both cases, but with opposite sign.

The expected value of $\langle \mathrm{Tr}\, M\rangle$ for large N is given by

$$\langle \mathrm{Tr}\, M\rangle = \sum_{j=0}^{N-1} \langle\lambda_j\rangle = \frac{1}{2\pi}\int \mathrm{d}p\, \mathrm{d}\lambda\, \lambda\theta[E_F - H(p,\lambda)]. \tag{22}$$

Calculations of higher moments $\langle \mathrm{Tr}\, M^n\rangle$ are carried out in the same manner. Upon rescaling, we find that $\langle \mathrm{Tr}\, M\rangle$ grows as $N^{3/2}$, and the scaled expectation value is given by

$$\mu = \lim_{N\to\infty} \frac{1}{N^{3/2}} \langle \mathrm{Tr}\, M\rangle = \frac{1}{2\pi}\int \mathrm{d}p\, \mathrm{d}\lambda\, \lambda\theta\, [\epsilon_F - H_{sc}(p,\lambda)] \tag{23}$$

which reduces to

$$\mu = \frac{1}{\pi}\int \mathrm{d}\lambda\, \lambda\sqrt{2\epsilon_F + 2g(\mathrm{i}\lambda)^p}\theta[2\epsilon_F + 2g(\mathrm{i}\lambda)^p]. \tag{24}$$

Using the same two-segment straight line path as before, we find that

$$\mu = -\mathrm{i}\left(\frac{\pi}{2g}\right)^{\frac{1}{p+2}} \frac{\cos\left(\frac{\pi}{p}\right)}{\sin\left(\frac{\pi}{p}\right)^{\frac{2}{p+2}}} \left[\frac{\Gamma(3/2+1/p)}{\Gamma(1+1/p)}\right]^{\frac{p+4}{p+2}} \frac{\Gamma(1+2/p)}{\Gamma(3/2+2/p)}. \tag{25}$$

For $p = 2$, $\mu = 0$, as expected for a harmonic oscillator. For $p > 2$, the expectation value μ is imaginary because $\langle\lambda_j\rangle$ for each eigenstate of the reduced problem is imaginary [1]. For $p = 3$, $\mu = -0.520\,06\mathrm{i}$. For $p = 4$, $\mu = -0.772\,539\mathrm{i}$. In the limit $p \to \infty$, μ goes to $-\mathrm{i}$. This behavior is easy to understand, because in this limit, the turning points become degenerate at $-\mathrm{i}$.

4. Special case of Tr M^4

For the case of a Tr M^4 interaction, we can explicitly exhibit the equivalence of the PT-symmetric matrix model with a conventional Hermitian quantum-mechanical system. As in the single-variable case, there is a parity-violating anomaly, in the form of an extra term in the Hermitian form of the Hamiltonian, proportional to $\hbar$. We show below that the anomaly term does not contribute at leading order in the large-N limit.

The derivation of the equivalence closely follows the path integral derivation for the single-variable case [9, 10]. The Euclidean Lagrangian is

$$L = \frac{1}{2}\,\mathrm{Tr}\left(\frac{\mathrm{d}M}{\mathrm{d}t}\right)^2 + \frac{1}{2}m^2\,\mathrm{Tr}\,M^2 - \frac{g}{N}\,\mathrm{Tr}\,M^4 \tag{26}$$

and the path integral expression for the partition function is

$$Z = \int [\mathrm{d}M]\exp\left\{-\int \mathrm{d}t\,L\right\}. \tag{27}$$

Motivated by the case of a single variable, we make the substitution

$$M = -2\mathrm{i}\sqrt{1+\mathrm{i}H} \tag{28}$$

where H is a Hermitian matrix. Because M and H are simultaneously diagonalizable, this transformation is tantamount to the relation

$$\lambda_j = -2\mathrm{i}\sqrt{1+\mathrm{i}h_j} \tag{29}$$

between the eigenvalues of M and the eigenvalues h_j of H. The change of variables induces a measure factor

$$[\mathrm{d}M] = \frac{[\mathrm{d}H]}{\mathrm{Det}[\sqrt{1+\mathrm{i}H}]} \tag{30}$$

where the functional determinant depends only on the eigenvalues of H. The Lagrangian becomes

$$L = \frac{1}{2}\,\mathrm{Tr}\frac{(\mathrm{d}H/\mathrm{d}t)^2}{1+\mathrm{i}H} - 2m^2\,\mathrm{Tr}\,(1+\mathrm{i}H) - 16\frac{g}{N}\,\mathrm{Tr}\,(1+\mathrm{i}H)^2 \tag{31}$$

at the classical level. However, following [10], we note that in the matrix case the change of variables introduces an extra term in the potential of the form

$$\Delta V = \sum_j \frac{1}{8}\left[\frac{\mathrm{d}}{\mathrm{d}h_j}\left(\frac{\mathrm{d}h_j}{\mathrm{d}\lambda_j}\right)\right]^2 \tag{32}$$

which can be written as

$$\Delta V = -\frac{1}{32}\sum_j \frac{1}{1+\mathrm{i}h_j} = -\frac{1}{32}\,\mathrm{Tr}\left(\frac{1}{1+\mathrm{i}H}\right). \tag{33}$$

The partition function is now

$$Z = \int \frac{[\mathrm{d}H]}{\det[\sqrt{1+\mathrm{i}H}]} \exp\left[-\int \mathrm{d}t L\right], \tag{34}$$

where

$$L = \frac{1}{2}\operatorname{Tr}\frac{(\mathrm{d}H/\mathrm{d}t)^2}{1+\mathrm{i}H} - 2m^2 \operatorname{Tr}(1+\mathrm{i}H) - \frac{16g}{N}\operatorname{Tr}(1+\mathrm{i}H)^2 - \frac{1}{32}\operatorname{Tr}\left(\frac{1}{1+\mathrm{i}H}\right). \tag{35}$$

We introduce a Hermitian matrix-valued field Π using the identity

$$\frac{1}{\det[\sqrt{1+\mathrm{i}H}]} = \int [\mathrm{d}\Pi] \exp\left\{-\int \mathrm{d}t \operatorname{Tr}\left[\frac{1}{2}(1+\mathrm{i}H)(\Pi-\Pi_0)^2\right]\right\}, \tag{36}$$

where $\Pi_0 = (\mathrm{i}\dot{H} + 1/4)/(1+\mathrm{i}H)$. Dropping and adding appropriate total derivatives and integrating by parts yields

$$Z = \int [\mathrm{d}H][\mathrm{d}\Pi] \exp\left[-\int \mathrm{d}t L'\right] \tag{37}$$

where

$$L' = \operatorname{Tr}\left[-2m^2(1+\mathrm{i}H) - 16\frac{g}{N}(1+\mathrm{i}H)^2 + \frac{1}{2}(1+\mathrm{i}H)\Pi^2\right] + \operatorname{Tr}\left[\dot{\Pi}(1+\mathrm{i}H) - \frac{1}{4}\Pi\right]. \tag{38}$$

The integration over H is Gaussian, and gives

$$Z = \int [\mathrm{d}\Pi] \exp\left\{-\int \mathrm{d}t \operatorname{Tr}\left[\frac{N}{64g}\left(\dot{\Pi}^2 - 2m^2\Pi^2 + \frac{1}{4}\Pi^4\right) - \frac{1}{4}\Pi\right]\right\}. \tag{39}$$

After the rescaling $\Pi \to \sqrt{32g/N}\,\Pi$ we have finally

$$Z = \int [\mathrm{d}\Pi] \exp\left\{-\int \mathrm{d}t \operatorname{Tr}\left[\frac{1}{2}(\dot{\Pi}^2 - 2m^2\Pi^2) + \frac{4g}{N}\Pi^4 - \sqrt{\frac{2g}{N}}\,\Pi\right]\right\}. \tag{40}$$

This proves the equivalence of the PT-symmetric matrix model defined by

$$L = \frac{1}{2}\operatorname{Tr}\left(\frac{\mathrm{d}M}{\mathrm{d}t}\right)^2 + \frac{1}{2}m^2 \operatorname{Tr} M^2 - \frac{g}{N}\operatorname{Tr} M^4 \tag{41}$$

to the conventional quantum mechanics matrix model given by

$$L' = \frac{1}{2}\operatorname{Tr}\left(\frac{\mathrm{d}\Pi}{\mathrm{d}t}\right)^2 - \sqrt{\frac{2g}{N}}\operatorname{Tr}\Pi - m^2\operatorname{Tr}\Pi^2 + \frac{4g}{N}\operatorname{Tr}\Pi^4. \tag{42}$$

This equivalence implies that the energy eigenvalues of the corresponding Hamiltonians are the same. This could also be proven using the single-variable equivalence for the special case of singlet states, but the functional integral proof encompasses both singlet and non-singlet states at once. The equivalence of these two models also allows for an easy proof of the singlet nature of the ground state. Standard variational arguments show that the ground state of the Hermitian form is a singlet. The direct quantum-mechanical equivalence of the single-variable case is then sufficient to prove that the ground state of the PT-symmetric form is also a singlet.

As in the single-variable case, there is a linear term of order $\hbar$ appearing in the Lagrangian and Hamiltonian of the Hermitian form of the model. This term represents a quantum-mechanical anomaly special to the Tr M^4 model. To determine the fate of the anomaly in the large-N limit, we construct the scaled Hamiltonian of the Hermitian form in exactly the same way as for the PT-symmetric form. It is given by

$$H_{sc} = \frac{1}{2}p^2 - \frac{1}{N}\sqrt{2g}x - m^2x^2 + 4gx^4, \tag{43}$$

indicating that the effect of the anomaly is absent in leading order of the large-N expansion. One easily checks for the $m = 0$ case that the Hermitian form without the linear term reproduces the PT-symmetric prediction for $E_\infty^{(0)}$ at $p = 4$.

5. $O(N)$ vector models

The analysis of the $O(N)$-invariant PT-symmetric model with a quartic interaction is similar to that of the Tr M^4 matrix model. Consider a model with Euclidean Lagrangian given by

$$L_E = \sum_{j=1}^{N} \left[\frac{1}{2} (\partial_t x_j)^2 + \frac{1}{2} m^2 x_j^2 - \lambda x_j^4 \right] - \frac{g}{N} \left(\sum_{j=1}^{N} x_j^2 \right)^2 \tag{44}$$

where g and λ are non-negative. When $g = 0$, we have N decoupled one-dimensional systems; for $\lambda = 0$, we have a model with $O(N)$ symmetry. When both g and λ are non-zero, the model has only an S_N permutation symmetry. From the standpoint of PT symmetry, the interaction terms can be considered as members of a family of PT-invariant interactions

$$-\lambda \sum_{j=1}^{N} (-\mathrm{i} x_j)^{2p} - \frac{g}{N} \left(-\sum_{j=1}^{N} x_j^2 \right)^q \tag{45}$$

which are invariant under PT symmetry. This class of models is well defined for $p = q = 1$, and must be defined for $p, q > 1$ by an appropriate analytic continuation of the x_j as necessary [1].

It is convenient to consider this model as a subset of a larger class of models, with a Lagrangian of the form

$$L_E = \sum_{j=1}^{N} \left[\frac{1}{2} (\partial_t x_j)^2 + \frac{1}{2} m^2 x_j^2 \right] - \sum_{j,k=1}^{N} x_j^2 \Lambda_{jk} x_k^2. \tag{46}$$

The classical stability of the potential for large x_j is governed by the eigenvalues of Λ. For the model of particular interest to us,

$$\Lambda = \lambda I + g P \tag{47}$$

where P is the one-dimensional projector

$$P = \frac{1}{N} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & \dots \\ 1 & \dots & \dots \end{pmatrix} \tag{48}$$

satisfying $P^2 = P$. The decomposition $\Lambda = \lambda(I - P) + (g + \lambda)P$ shows that Λ has one eigenvalue $g + \lambda$ and $N - 1$ eigenvalues with value λ. The eigenvalue $g + \lambda$ is associated with variations in $\vec{x}^2$, i.e., variations in the radial direction.

6. Equivalence of PT-symmetric vector models to Hermitian models

We will analyze the case where all eigenvalues of Λ are positive using functional integration. With the substitution

$$x_j \to -2\mathrm{i}\sqrt{c_j + \mathrm{i}\psi_j} \tag{49}$$

familiar from the one-component case, L_E becomes

$$L_E = \sum_j \left[\frac{1}{2} \frac{(\partial_t \psi_j)^2}{(c_j + \mathrm{i}\psi_j)} - 2m^2 (c_j + \mathrm{i}\psi_j) \right] - 16 \sum_{jk} \Lambda_{jk} (c_j + \mathrm{i}\psi_j)(c_k + \mathrm{i}\psi_k). \tag{50}$$

The generating function for the model is given by

$$Z = \int \prod_j \frac{[\mathrm{d}\psi_j]}{\sqrt{\det(c_j + \mathrm{i}\psi_j)}} \exp \left[-\int \mathrm{d}t \left[L_E - \sum_j \frac{1}{32} \left(\frac{1}{c_j + \mathrm{i}\psi_j} \right) \right] \right] \tag{51}$$

where the change of variables has generated both a functional determinant and additional term, formally of order $\hbar^2$, in the action. As pointed out in [10], both terms are required to obtain correct results in the functional integral formalism.

The functional determinant may be written as

$$\prod_j \frac{1}{\det[\sqrt{c_j + \mathrm{i}\psi_j}]} = \int \prod_j [\mathrm{d}h_j] \exp\left\{-\int \mathrm{d}t \left[\frac{1}{2}(c_j + \mathrm{i}\psi_j)\left(h_j - \frac{\mathrm{i}\dot{\psi}_j + 1/4}{c_j + \mathrm{i}\psi_j}\right)^2\right]\right\} \tag{52}$$

which introduces a new set of fields h_j. The derivation proceeds as in the single-variable case. After integration by parts on the $h_j\dot{\psi}_j$ terms, and adding and subtracting total derivatives, the functional integral over the ψ_j fields can be carried out exactly. The integral is both local and quadratic, and requires that the matrix Λ have positive eigenvalues for convergence. The result of this integration is

$$Z = \int \prod_n [\mathrm{d}h_n] \exp\left[-\int \mathrm{d}t\, L_H\right] \tag{53}$$

where L_H is given by

$$L_H = \frac{1}{64}\sum_{jk} \Lambda^{-1}_{jk}\left[\frac{1}{2}h_j^2 + \dot{h}_j - 2m^2\right]\left[\frac{1}{2}h_k^2 + \dot{h}_k - 2m^2\right] - \sum_j \frac{1}{4}h_j. \tag{54}$$

After discarding total derivatives, we obtain

$$L_H = \frac{1}{64}\sum_{jk} \Lambda^{-1}_{jk}\left[\dot{h}_j\dot{h}_k + \frac{1}{4}\left(h_j^2 - 4m^2\right)\left(h_k^2 - 4m^2\right)\right] - \sum_j \frac{1}{4}h_j \tag{55}$$

which gives the Hermitian form for our general PT-symmetric model with N fields.

In the particular case we are interested in, we have

$$\Lambda^{-1} = \frac{1}{\lambda}(I - P) + \frac{1}{g + \lambda}P. \tag{56}$$

The Lagrangian may be written as

$$\begin{aligned} L_E = {} & \frac{1}{64\lambda}\sum_j \left[\dot{h}_j^2 + \frac{1}{4}\left(h_j^2 - 4m^2\right)^2\right] - \frac{1}{4}\sum_j h_j \\ & - \frac{g}{64N\lambda(g+\lambda)}\left[\left(\sum_j \dot{h}_j\right)^2 + \frac{1}{4}\left(\sum_j \left(h_j^2 - 4m^2\right)\right)^2\right]. \end{aligned} \tag{57}$$

It is helpful to immediately rescale all the fields as $h_j \to \sqrt{32\lambda}h_j$:

$$\begin{aligned} L_E = {} & \sum_j \left[\frac{1}{2}\dot{h}_j^2 + 4\lambda\left(h_j^2 - \frac{m^2}{8\lambda}\right)^2\right] - \sqrt{2\lambda}\sum_j h_j \\ & - \frac{g}{N(g+\lambda)}\left[\frac{1}{2}\left(\sum_j \dot{h}_j\right)^2 + 4\lambda\left(\sum_j \left(h_j^2 - \frac{m^2}{8\lambda}\right)\right)^2\right]. \end{aligned} \tag{58}$$

At this point, the S_N permutation symmetry is still manifest, and it is clear that the field $\sum_j h_j$ plays a special role.

In order to understand the strategy for rewriting the model in a form in which the limit $\lambda \to 0$ can easily be taken, it is useful to work out explicitly the case of $N = 2$ first. It is apparent that a rotation of the fields will be desirable. We define suggestively new fields σ and π given by

$$h_1 = \tfrac{1}{\sqrt{2}}(\sigma + \pi), \qquad h_2 = \tfrac{1}{\sqrt{2}}(\sigma - \pi). \tag{59}$$

After some algebra and the rescaling

$$\sigma \to \sqrt{\frac{g+\lambda}{\lambda}}\sigma \tag{60}$$

we arrive at

$$\begin{aligned} L_E = {}& \frac{1}{2}\dot{\sigma}^2 + \frac{1}{2}\dot{\pi}^2 - m^2\sigma^2 - \frac{\lambda m^2}{g+\lambda}\pi^2 + 2(g+\lambda)\sigma^4 + \frac{2\lambda^2}{g+\lambda}\pi^4 \\ & + (8g + 12\lambda)\sigma^2\pi^2 - 2\sqrt{g+\lambda}\sigma. \end{aligned} \tag{61}$$

Note the natural hierarchy between the masses for $\lambda \ll g$. The $O(2)$ symmetric limit of the original PT-symmetric model is obtained in the limit $\lambda \to 0$, where we have

$$L_E = \tfrac{1}{2}\dot{\sigma}^2 + \tfrac{1}{2}\dot{\pi}^2 - m^2\sigma^2 + 2g\sigma^4 + 8g\sigma^2\pi^2 - 2\sqrt{g}\sigma. \tag{62}$$

The field π has no mass term, indicating its relation to the angular degrees of freedom in the original Lagrangian. However, radiative corrections generate a mass for the π field via the $\sigma^2\pi^2$ interaction. As in the one-component case, there is a linear anomaly term, but only for σ.

We now turn to the more difficult case of the $\lambda \to 0$ limit for arbitrary N. As before, we introduce a field σ defined by

$$\sigma = \frac{1}{\sqrt{N}}\sum_j h_j \tag{63}$$

as well as a set of $N - 1$ fields π_k with $k = 1, \ldots, N - 1$ related to the h_j fields by a rotation so that $\sigma^2 + \vec{\pi}^2 = \vec{h}^2$. Each field h_j can be written as

$$h_j = \frac{1}{\sqrt{N}}\sigma + \tilde{h}_j \tag{64}$$

where $\sum_j \tilde{h}_j = 0$. This property is crucial in eliminating a term in L_E which diverges as $\lambda^{-1/2}$ as $\lambda \to 0$. The Lagrangian now can be written as

$$\begin{aligned} L_E = {}& \frac{1}{2}\dot{\sigma}^2 + \sum_j \frac{1}{2}\dot{\pi}_j^2 - m^2(\sigma^2 + \vec{\pi}^2) + 4\lambda\sum_j h_j^4 \\ & + \frac{4}{N}\left(\frac{\lambda^2}{g+\lambda} - \lambda\right)\left(\sigma^2 + \vec{\pi}^2 - \frac{Nm^2}{8\lambda}\right)^2 - \sqrt{2\lambda N}\sigma. \end{aligned} \tag{65}$$

The rescaling $\sigma \to \sqrt{(g+\lambda)/\lambda}\sigma$ plus some careful algebra yields the $\lambda \to 0$ limit as

$$L_E = \frac{1}{2}\dot{\sigma}^2 + \frac{1}{2}\dot{\vec{\pi}}^2 - m^2\sigma^2 + \frac{4g}{N}\sigma^4 + \frac{16g}{N}\sigma^2\vec{\pi}^2 - \sqrt{2gN}\sigma \tag{66}$$

which agrees with our previous result for $N = 2$, and agrees with the known result for a single degree of freedom if we take $N = 1$ and drop the $\vec{\pi}$ field altogether. This is a Hermitian form of the PT-symmetric anharmonic oscillator with $O(N)$ symmetry, derived as the limit of a PT-symmetric model with S_N symmetry. The Hermitian form has several novel features. Note that both the S_N and $O(N)$ symmetries are no longer manifest, but there is an explicit $O(N - 1)$ symmetry associated with rotations of the $\vec{\pi}$ field. As in the $N = 2$ case, there is no mass term for the $\vec{\pi}$ field. Furthermore, there is no $(\vec{\pi}^2)^2$ term, although there is a $\vec{\pi}^2\sigma^2$

interaction. The anomaly term again involves only σ, and breaks the symmetry $\sigma \to -\sigma$ possessed by the rest of the Lagrangian. Analyzing the Lagrangian at the classical level, we see that if $m^2 > 0$, the σ field is moving in a double-well potential, perturbed by the anomaly so that $\langle\sigma\rangle > 0$. On the other hand, if $m^2 < 0$, σ moves in a single-well anharmonic oscillator, again with the linear anomaly term making $\langle\sigma\rangle > 0$. In either case, the $\vec{\pi}^2\sigma^2$ interaction will generate a mass for the $\vec{\pi}$ field. All of this is consistent with the association of σ and $\vec{\pi}$ with the radial and angular degrees of freedom, respectively, in the original PT-symmetric model. This equivalence between PT-symmetric and Hermitian forms may be compared to the results of [11], where a somewhat different equivalence is derived. In that work, the generating function for a Hermitian x^4 theory is shown to be equivalent to a sum over generating functions for a class of single-component PT-symmetric models, with each element of the class representing a different angular momentum. As we discuss below in the context of the large-N limit, both approaches lead to an anomaly term with a linear dependence on the angular momentum quantum number l.

7. Large-N limit of vector models

We will defer a more detailed discussion of this model for finite N, and turn to its large-N limit. One more rescaling $\sigma \to \sqrt{N}\sigma$ gives the Lagrangian

$$L_E = \frac{N}{2}\dot{\sigma}^2 + \frac{1}{2}\dot{\vec{\pi}}^2 - Nm^2\sigma^2 + 4gN\sigma^4 + 16g\sigma^2\vec{\pi}^2 - N\sqrt{2g}\sigma. \tag{67}$$

We see that the anomaly term survives in the large-N limit, unlike the matrix model case [4]. After integrating over the $N-1$ components of the $\vec{\pi}$ field, we have the large-N effective potential V_{eff} for σ:

$$V_{\text{eff}}/N = -m^2\sigma^2 + 4g\sigma^4 + \tfrac{1}{2}\sqrt{32g\sigma^2} - \sqrt{2g}\sigma. \tag{68}$$

It is striking that the anomaly term has virtually the same form as the zero-point energy of the $\vec{\pi}$ field. The anomaly term breaks the discrete $\sigma \to -\sigma$ symmetry of the other terms of the Lagrangian, and always favors $\sigma \geqslant 0$. The effective potential has a global minimum with σ positive for $m^2 > 32^{1/3}g^{2/3}$. For $m^2 < 32^{1/3}g^{2/3}$, there does not appear to be a stable solution with $\sigma > 0$, and $\sigma = 0$ is the stable solution to leading order in the $1/N$ expansion. This change in the behavior of the effective potential as m^2 is varied is not seen in the corresponding Hermitian model [14], and indicates a need for care in analyzing the model. Based on our preliminary analysis of the Hermitian form for finite N, we believe that this behavior is associated with the large-N limit, and does not indicate a fundamental restriction on m^2.

The large-N effective potential was derived from a Lagrangian with unusual properties, associated with the Hermitian form of the original model. It is therefore surprising that, once the form of the large-N effective potential is known, it can be derived heuristically in a more conventional way. We start from the $O(N)$-symmetric Lagrangian

$$L_E = \sum_{j=1}^{N}\left[\frac{1}{2}(\partial_t x_j)^2 + \frac{1}{2}m^2 x_j^2\right] - \frac{g}{N}\left(\sum_{j=1}^{N} x_j^2\right)^2 \tag{69}$$

and add a quadratic term in a constraint field ρ

$$L_E \to L_E + \frac{g}{N}\left(\frac{2N\rho}{g} + \sum_{j=1}^{N} x_j^2 - \frac{Nm^2}{4g}\right)^2 \tag{70}$$

yielding

$$L_E = \sum_{j=1}^{N} \left[\frac{1}{2} (\partial_t x_j)^2 + 4\rho x_j^2 \right] + \frac{4N\rho^2}{g} - \frac{Nm^2\rho}{g} + \frac{Nm^4}{16g}. \tag{71}$$

If we integrate over x_j in a completely conventional way, we obtain the large-N effective potential

$$V_{\text{eff}}/N = \frac{4\rho^2}{g} - \frac{m^2\rho}{g} + \sqrt{2\rho} + \frac{m^4}{16g}. \tag{72}$$

This is essentially identical to our previous expression after identifying $\rho = g\sigma^2$. However, we lack a fundmental justification for this approach. We know that great care must be taken in specifying the contour of integration in typical PT-symmetric models, yet the x_j fields were integrated over quite conventionally. If this approach has validity, it seems likely that the choice of integration contours for ρ and $\vec{x}$ is crucial. However, only the saddle point matters to leading order in $1/N$, so it is possible for this heuristic derivation to be correct even though we lack a direct, complete treatment of the original PT-symmetric model.

There is another approach to the effective potential that sheds some light on the role of angular momentum in PT-symmetric vector models. Let us take as our starting point the Hamiltonian for the PT-symmetric vector model after the introduction of the constraint field:

$$H = \sum_{j=1}^{N} \left[\frac{1}{2} p_j^2 + 4\rho x_j^2 \right] + \frac{4N\rho^2}{g} - \frac{Nm^2\rho}{g} \tag{73}$$

where for simplicity we have dropped the m^4 constant term. The reduced Hamiltonian for the radial degree of freedom can be written as

$$H = -\frac{1}{2} \frac{\partial^2}{\partial r^2} + \frac{(N + 2l - 1)(N + 2l - 3)}{8r^2} + 4\rho r^2 + \frac{4N\rho^2}{g} - \frac{Nm^2\rho}{g} \tag{74}$$

where l is a non-negative integer [13]. Rescaling the radial coordinate $r \to N^{1/2} r$ leads to a potential proportional to N which is a function of both r and ρ and a kinetic term which is of order $1/N$. It is easy to minimize the potential as a function of r; the final result, after setting $l = 0$, is identical to the expression for V_{eff}/N as a function of ρ in the large-N limit. Thus we see that this radial formalism yields results for the ground-state energy equivalent to other approaches at leading order in the large-N expansion. Alternatively, one can take the angular momentum quantum number l to be of order N. Elimination of the r variable then leads to an effective potential of the form

$$V_{\text{eff}} = (N + 2l)\sqrt{2\rho} + \frac{4N\rho^2}{g} - \frac{Nm^2\rho}{g} \tag{75}$$

which displays the l-dependent anomaly term first observed in [11]. It is also possible to show the equivalence of the radial formalism directly, without introducing the composite field ρ. Note that the radial approach demonstrates that the angular momentum term makes a positive contribution to the ground-state energy in the PT-symmetric case, exactly as it does in the Hermitian case.

8. PT-symmetric field theory

If we boldly apply the constraint field approach to a PT-symmetric field theory with a $-g(\vec{\phi}^2)^2$ interaction in d dimensions, we obtain the effective potential

$$V_{\text{eff}}/N = \frac{4\rho^2}{g} - \frac{m^2\rho}{g} + \frac{m^4}{16g} + \frac{1}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \ln[k^2 + 8\rho]. \tag{76}$$

Models of this type were rejected decades ago [14] because of stability concerns at both the classical and quantum levels, although there were early indications that such theories were in fact sensible [15]. Within the framework of PT-symmetric models, such stability issues cannot be addressed without a detailed understanding of the contours used in functional integration. However, it is straightforward to check that renormalization of g in $d = 4$ gives an asymptotically free theory, with beta function $\beta = -g^2/2\pi^2$ in the large-N limit. If PT-symmetric scalar field theories exist in four dimensions and are indeed asymptotically free, the possible implications for particle physics are large, and provide ample justification for further work.

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